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Hyperbolic Complex Numbers and Nonlinear Sigma Models

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We show that the hyperbolic complex numbers or double numbers can be used to generate solutions of two-dimensional Minkowskian sigma models with values on noncompact manifolds.

1. INTRODUCTION

Two-dimensional sigma models are interesting objects for both physicists and mathematicians. In fact, they are nice examples of integrable systems with dual symmetry (Eichenherr and Forger, 1979) and infinitely many conservation laws (Forger, 1983). Furthermore, they provide a good theoretical laboratory to investigate the four-dimensional gauge theories (such as Yang–Mills systems). Mathematically, these sigma models can be defined as harmonic maps (Fujii, 1985) from a two-dimensional space to certain manifolds (Riemannian symmetric spaces, . . .). Many methods have been developed in order to construct explicit solutions for the two-dimensional sigma models. These methods were initiated by the work of Pohlmeyer (1976), Zakharov and Mikhalov (1978), and Eichenherr and Forger (1980). More precisely, a method of construction of multisoliton solutions has been introduced by Saint-Aubin (1982), Harnad *et al.* (1984a) for sigma models with values on Riemannian symmetric spaces and has been extended to general integrable systems (Harnad *et al.*, 1984b). For the particular case of the $\mathbf{CP}^n$ model the so-called "holomorphic" method was introduced by Borchers and Garber (1980) and systematically developed by Zakrzewski (1982, 1984). More recently, Antoine and Piette (1986) have

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reexamined, in the light of a new geometrical result, the method of construction of solutions for classical sigma models on (real, complex, and quaternionic) Grassmann manifolds of compact or noncompact case.

The above methods intensively use usual complex analysis. In particular, the Zakrzewski (1982, 1984) method is based on the construction of a basis of analytic vectors. In this paper we show that the so-called hyperbolic complex numbers and hyperbolic analytic functions can be used to construct particular solutions of two-dimensional sigma models with values on noncompact spaces.

The hyperbolic complex numbers or double numbers have been extensively studied by mathematicians in the framework of pseudo-Euclidean geometries (Yaglom, 1968), Cayley-Dickson algebras (Polubarinov, 1985), and Clifford algebras (Salingaros, 1981). They were introduced by Clifford (1968), who called them "motors" because he was concerned with the use of these numbers in mechanics . Muses (1970) was the first to point out the physical usefulness of hyperbolic numbers. Recently these numbers have been introduced by Kunstatter *et al.* (1983) in a geometric interpretation of a generalized theory of gravitation and by Lambert and Kibler (1986) in order to generate nonbijective canonical transformations.

2. HYPERBOLIC COMPLEX ALGEBRA AND ANALYSIS

Let j denote the so-called hyperbolic imaginary unit defined by $j^2 = +1$. Then the set of hyperbolic complex numbers Ω can be defined by

$$\Omega = \{z = x + jy;\ x, y \in \mathbb{R}\} \cong \mathbb{R} \oplus \mathbb{R}$$

The conjugation number of $z = x + jy$ is $\tilde{z} = x - jy$ and the norm of z is the real number $|z|$ such that $|z|^2 = x^2 - y^2$. When $|z|^2 \neq 0$, it is possible to define a unique inverse $z^{-1} = \tilde{z}/|z|^2$. The numbers z such that $|z|^2 = 0$ and $z \neq 0$ are called zero divisors and lead to interesting algebraic problems (Demys, 1987). Because of these zero divisors Ω is an (Abelian) ring and not a field.

For those preferring not to work with the algebraic unit j it is possible to construct 2×2 matrix representation of Ω over $\mathbb{R}$. This is achieved by using the following identifications:

$$1 \leftrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad j \leftrightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Hence the hyperbolic complex number $z = x + jy$ can be represented by the matrix

$$Z = \begin{bmatrix} x & y \\ y & x \end{bmatrix}$$

Let $u(x, y)$ and $v(x, y)$ be differentiable functions defined on a domain D of $\mathbb{R}^2$. It is natural to introduce the function $f(z) = u(x, y) + jv(x, y)$. This function is said to be hyperbolic analytic (or h-analytic) if and only if the limit

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

has the same value irrespective of the path along which h tends to zero (and $h\tilde{h} \neq 0$). If u and v are sufficiently derivable, then the necessary and sufficient conditions for the function $f(z)$ to be h-analytic are

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y}; \qquad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \tag{1}$$

which are the hyperbolic analogs of the Cauchy–Riemann equations. It is obvious to check from equations (1) that the relations

$$\Box u = 0, \qquad \Box v = 0; \qquad \Box = \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \tag{2}$$

hold. Furthermore, equations (2) enable us to write u and v as

$$u = F(x+y) + G(x-y), \qquad v = F(x+y) - G(x-y) \tag{3}$$

where F and G are sufficiently differentiable arbitrary functions of x and y in D. It is interesting to note that u and v are constant along the lines $x = \pm y + c$, where c is an arbitrary real constant. These lines are the characteristics of the system defined by equations (1) and when $c = 0$, they reduce to the set of zero divisors of Ω. From equations (3) and identifying x with a time variable, it is then possible to consider any h-analytic function as a superposition of two waves (Laurentiev and Chabat, 1980).

Due to the fact that Ω is a ring, it is natural to endow the Cartesian product Ω^n with the structure of an Ω-module.

3. HYPERBOLIC COMPLEX GRASSMANNIAN MANIFOLDS

Let V be a vector of the n-dimensional Ω-module Ω^n. As in the usual complex case, we introduce the vector $V^{\#}$ defined by $V^{\#} = \tilde{V}^{\mathrm{tr}}$. Hence we are able to endow Ω^n with an indefinite norm.

We define the norm $\| V \|$ of V by

$$\| V \|^2 = V^{\#} V = V V^{\#} \tag{4}$$

Let us now consider the linear mapping $T: \Omega^n \to \Omega^n$ such that

$$\| TV \|^2 = \| V \|^2 \tag{5}$$

From equations (4) and (5) we easily check that

$$T^{\#}T = TT^{\#} = \mathbb{1} \tag{6}$$

where $\mathbb{1}$ is the $n \times n$ identity matrix. Following Zhong (1984), we call such a mapping a hyperbolic complex unitary transformation. All the T's form a group denoted by $U(n, \Omega)$. Locally we get the following isomorphism (Zhong, 1984, 1985):

$$U(n, \Omega) \cong GL(n, \mathbb{R}) \tag{7}$$

Let us introduce the manifold $G(N, n; \Omega)$ defined by

$$G(N, n; \Omega) = U(N, \Omega)/U(N-n, \Omega) \times U(n, \Omega) \tag{8}$$

Referring to the usual complex case, we call it a hyperbolic complex Grassmannian manifold (HCGM).

When $n = 1$, the HCGM can be geometrically characterized by the following propositions:

Proposition 1. The manifold $G(N, 1; \Omega)$ is locally isomorphic to the symmetric space

$$SL(N, \mathbb{R})/SL(N-1, \mathbb{R}) \times SO(1, 1)$$

Proof. We know that $GL(1, \mathbb{R})$ is locally isomorphic to $SO(1, 1)$. Then Proposition 1 follows immediately from equation (7).

Let X be a nonisotropic vector of Ω (i.e., a vector of nonzero norm) and let $\Omega(N)$ be the set of all $N \times N$ hyperbolic complex matrices. Then let us consider the matrix $P(X)$ of $\Omega(N)$ defined by

$$P(X) = XX^{\#}/\|X\|^2 \tag{9}$$

It is easy to check that $P(X)^2 = P(X)$, $P(X)^{\#} = P(X)$, and $\mathrm{Tr}(P(X)) = 1$. Hence $P(X)$ is a projector.

We note that if λ is not a zero divisor of Ω, the equation

$$P(\lambda X) = P(X) \tag{10}$$

holds. Therefore, with each projector $P(X)$ constructed with an arbitrary nonisotropic vector X we associate the nonisotropic part $D_0(X)$ of the straight line $D(X)$ defined by

$$D(X) = \{Y \in \Omega^N \colon Y = \lambda X;\ \lambda \in \Omega\} \tag{11}$$

Let $H(m, n)$ be the $(m+n-1)$-dimensional real hyperboloid defined by the equation

$$\sum_{j=1}^{m} x_j^2 - \sum_{j=m+1}^{m+n} x_j^2 = 1; \qquad x_j \in \mathbb{R}$$

Then we get the following result.

Proposition 2. The set of all projectors $P(X)$ defined by equation (9) is isomorphic to the coset

$$M = H(N, N)/H(1, 1) \tag{12}$$

Proof. Each $P(X)$ can be associated with $D_0(X)$. The set of all $D_0(X)$'s can be viewed as the coset $(H_+ \cup H_-)/\sim$. The manifold H_+ (resp. H_-) is the set of all vectors of norm $+1$ (resp. -1) and $\sim$ is the equivalence, which identifies the vectors of norm either $+1$ or -1 belonging to the same $D_0(X)$. From (4) we get the following isomorphisms: $H_+ \cong H_- \cong H(N, N)$. In order to identify the vectors of $D_0(X)$ having the norm either $+1$ or -1, we use elements of the noncompact group $O(1, 1)$. The latter is isomorphic to $H(1, 1) \times \mathbf{Z}_2$. Hence we get the following isomorphism

$$(H_+ \cup H_-)/\sim \cong H(N, N)/H(1, 1)$$

Remark. From (12) it is easy to see that the manifold M is isomorphic to the symmetric space

$$SO(N, N)/SO(N-1, N) \times SO(1, 1) \tag{13}$$

which is homeomorphic to $\mathbb{R}^{N-1} \times \mathbb{R}P^{N-1}$ (where $\mathbb{R}P^n$ denotes the n-dimensional real projective space).

From (5) we see that the group $U(N; \Omega)$ acts transitively on the hypersurface H_+ (resp. H_-) defined above. Furthermore, it is possible to check that $U(N-1; \Omega)$ is conjugated to each isotropy group of an arbitrary point of H_+ (resp. H_-). Hence we get the following isomorphisms: $H_+ \cong H_- \cong U(N; \Omega)/U(N-1; \Omega)$. Equation (6) leads to $U(1, \Omega) \cong SO(1, 1)$. Then the coset $H_+ \cup H_-/\sim$ happens to be isomorphic to $U(N; \Omega)/U(N-1; \Omega) \times U(1; \Omega)$, i.e., $G(N, 1; \Omega)$. From this we get the following:

Proposition 3. The set of all projectors $P(X)$ defined by equation (9) is isomorphic to the HCGM $G(N, 1; \Omega)$.

In the usual complex case, the set of all straight lines passing through the origin in the vector space $\mathbf{C}^{N+1}$ gives rise to the projective space $\mathbf{CP}^N$. From the results of this section we identify $G(N, 1; \Omega)$ with the set of all non-isotropic parts of straight lines of the module Ω^N. Thus, $G(N, 1; \Omega)$ happens to be the hyperbolic analog of the complex projective space and can be properly defined in the framework of the projective geometry on ring.

4. SIGMA MODELS ON HCGM

Let $V: \Omega \to \Omega^N$ be an arbitrary mapping. We define $V(z)$ as an h-analytic vector if and only if each component of $V(z)$ is an h-analytic function.

Let us consider an $(N-1)$-dimensional h-analytic vector $V(z)$. We define the map $P:\Omega \to \Omega(N)$ by

$$P(z)=\frac{1}{1+\| V(z)\|^2}\begin{bmatrix} 1 & V(z)^{\#} \\ V(z) & V(z)V(z)^{\#} \end{bmatrix} \tag{14}$$

It is easy to see that $P(z)$ defines a projector. As long as z does not belong to the singular curve C: $1+\| V(z)\|^2=0$, it follows from Proposition 3 that $P(z)$ can be associated with a point of $G(N, 1; \Omega)$. The main property of the map P is given by the following result:

Proposition 4. The projector $P(z)$ satisfies the equation

$$[P(z), \Box P(z)]=0 \tag{15}$$

where $\Box$ is defined by (2).

Proof. The result comes from the generalization of the Zakrzewski (1982, 1984) method to the h-analytic functions (see Piette and Lambert, 1987).

Equation (15) appears to be the hyperbolic complex analog of the field equation of the two-dimensional Minkowskian sigma model with values on a Grassmannian manifold. Therefore $P(z)$ can be considered as a solution of the sigma model defined on Ω [i.e., $\mathbb{R}^2$ endowed with the metric diag$(1, -1)$ and with values on $G(N, 1; \Omega)$, i.e., $SO(N, N)/SO(N-1, N)\times SO(1, 1)$].

From a variational approach, (15) can be derived from a Lagrangian density $L(P)$ given by

$$L(P)=\mathrm{Tr}(\partial_\mu P\, \partial^\mu P)=\mathrm{Tr}\left[\left(\frac{\partial P}{\partial x}\right)^2-\left(\frac{\partial P}{\partial y}\right)^2\right] \tag{16}$$

Let us now illustrate the results of this section by two particular examples.

Example 1. When $N=2$ and $n=1$, the HCGM $G(N, n; \Omega)$ is isomorphic to $SO(1,2)/SO(1,1)=H(2,1)$. We note that the sigma model with values on $H(2,1)$ appears to be connected with the classical theory of relativistic strings (Nesterenko, 1987). Let us choose $V(z)=z-a$, where a is an arbitrary real constant. By a straightforward calculation we get from (1) that $V(z)$ is an h-analytic function. The projector $P(z)$ is given by

$$P(z)=\frac{1}{1+(x-a)^2-y^2}\begin{bmatrix} 1 & (x-a)-jy \\ (x-a)+jy & (x-a)^2-y^2 \end{bmatrix} \tag{17}$$

We check that (15) holds for this particular $P(z)$. Then (17) provides a solution of the Minkowskian sigma model with values on $SO(1,2)/SO(1,1)$.

The projector $P(z)$ and the associated Lagrangian density

$$L(P)=4/[1+(x-a)^2-y^2]^2$$

are singular along the same hyperbola: $y^2-(x-a^2)=1$.

Example 2. When $N=3$ and $n=1$, $G(N,n;\Omega)$ is isomorphic to $SO(3,3)/SO(2,3)\times SO(1,1)$. Let $V(z)$ be the two-dimensional h-analytic vector defined by

$$V(z)^{\mathrm{tr}}=(z+1,\, z^2-2).$$

From (14) we easily construct the 3×3 matrix $P(z)$. By direct computation it is possible to check that (15) holds. As in the preceding example, $P(z)$ and the Lagrangian density

$$L(P)=\frac{4[(y^2-x^2)^2-4(x+1)(y^2-x^2-2x)+5]}{[(4x^2+1)y^2-(x+1)^2-(x^2+y^2-2)^2-1]^2}$$

turn out to be singular along the same curve.

5. CONCLUSION

The main purpose of this paper is to show that the ring of hyperbolic complex numbers can be used to construct explicit solutions of some nonlinear sigma models with values on noncompact manifolds. As a matter of example we give only one explicit solution for the sigma model with values on the HCGM $G(N,1;\Omega)$ (This can be interpreted as the two-dimensional Minkowskian sigma model with values on $SO(N,N)/SO(N-1,N)\times SO(1,1)$]. Nevertheless, the results of the paper, mainly Proposition 4, can be extended to more general HCGM. This will be done in another paper.

It would be interesting to study the deep connection between the particular HCGM $G(N,1;\Omega)$ and projective geometries on rings. We know indeed on one hand that projective geometries on Clifford algebras are interesting objects for both physicists and mathematicians and on the other hand that Ω is the simplest noncompact Clifford algebra $C(1,0)$ (Salingaros, 1981).

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